THE ORIENTED GRAPH OF GRAFTINGS IN THE FUCHSIAN CASE

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Abstract. We prove the connectedness and calculate the diameter of the oriented graph of graftings associated to exotic $\mathbb{CP}^1$-structures on a compact surface S with a given holonomy representation of Fuchsian type.

1. Introduction

Let Γ_g be the fundamental group of a compact orientable surface S of genus $g \geq 2$, and $\rho : \Gamma_g \to \mathrm{PSL}(2, \mathbb{R})$ be a Fuchsian representation (namely discrete and faithful). Define an oriented graph $MG(\rho)$, the *graph of multi-graftings*, whose vertices are the equivalence classes of marked $\mathbb{CP}^1$-structures on S with holonomy ρ modulo projective diffeomorphims preserving the marking and whose edges are the couples (σ_1, σ_2), where σ_2 is a $\mathbb{CP}^1$-structure obtained from σ_1 by perfoming a 2π-grafting along a multi-curve modulo some equivalence class which is detailed in 2.4. The uniformizing structure for ρ is the one obtained by the quotient of $\mathbb{H}$ by the image of ρ. Every other $\mathbb{CP}^1$-structure is said to be exotic. Goldman proved that any $\mathbb{CP}^1$-structure with holonomy ρ is obtained from the uniformizing one by grafting a collection of disjoint simple closed curves (see [3]). As a corollary, $MG(\rho)$ is connected as a *non oriented* graph, and moreover its has diameter 2. In this paper we prove that the graph is still connected if we remove the uniformizing structure and that its oriented diameter is still 2.

Theorem 1.1. *Let σ_1 and σ_2 be two exotic projective structures sharing the same Fuchsian holonomy. Then σ_2 can be obtained from σ_1 by a sequence of two multi-graftings.*

A consequence of this result is that the fundamental group of $MG(\rho)$ is not finitely generated as soon as ρ is Fuchsian. Another consequence is that there exist positive cycles of graftings, namely finite sequences of marked $\mathbb{CP}^1$-structures $\sigma_0, \ldots, \sigma_r = \sigma_0$ such that for each $i = 1, \ldots, r$, σ_i is a grafting of σ_{i-1}. The integer r is then called the period of the cycle. Observe that an immediate corollary of the theorem is that any couple of exotic $\mathbb{CP}^1$-structures are contained in such a positive cycle of period bounded by 4. We will see that indeed there are such cycles of period 2.

To prove the results we will use some surgery operations on multicurves introduced by Luo [5] and later developed by Ito [4] and Thompson [6]. Our results and methods are closely related to Thompson's, but he considers the graph of graftings in the case of Schottky representations.

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2. Graftable curves

A marked surface of genus g is a closed compact oriented surface S of genus g together with the data of a universal cover $\pi : \widetilde{S} \to S$ of S and an identification of its covering group with Γ_g.

A marked projective structure (or a $\mathbb{CP}^1$-structure) is the data of a marked surface S together with a maximal atlas σ of charts from open sets of S to the Riemann sphere, such that the changes of coordinates are Moebius maps preserving the orientation. Two marked projective structures (S_i, σ_i) are considered as equivalent if there exists a projective diffeomorphism between S_1 and S_2 sending σ_1 to σ_2 and such that it lifts to the universal covers as a Γ_g-equivariant diffeomorphism.

For any projective structure we can construct a developing map $D : \widetilde{S} \to \mathbb{CP}^1$ by continuing a local chart of the projective structure along paths in S by appropriately chosing the charts of the given atlas. It is an equivariant map with respect to some representation $\rho : \Gamma_g \to \mathrm{PSL}(2, \mathbb{C})$.

2.1. **Definition.** Recall that a *multicurve* on a surface S is a finite disjoint union of simple closed curves none of which is homotopically trivial. Let σ be a marked projective structure on a compact orientable surface S. A multi-curve is said *graftable* (in σ) if all of its components have loxodromic holonomy and the developing map is injective when restricted to a lift of any of those components in $\widetilde{S}$. The condition is independent of the choice of developing map.

2.2. **Grafting along graftable curves.** If $\alpha = \{\alpha_i\}_{i\in I}$ is a graftable multi-curve, one can produce another marked projective structure, called the grafting along α, and denoted $\mathrm{Gr}(\sigma, \alpha)$. We recall the construction here. We cut the surface $\widetilde{S}$ along the lifts $\widetilde{\alpha_i}$'s of the curves α_i's, and glue to each of them a copy of $\mathbb{CP}^1 \setminus \overline{D(\widetilde{\alpha_i})}$ using the developing map for the gluing. We then obtain a new surface denoted by $\widetilde{S}'$, together with a new map $D' : \widetilde{S}' \to \mathbb{CP}^1$ which is defined by D on $\widetilde{S} \setminus \pi^{-1}(\cup_i \alpha_i)$ and by the identity on the spherical domains $\mathbb{CP}^1 \setminus \overline{D(\widetilde{\alpha_i})}$. The Γ_g-action on $\widetilde{S}$ induces a Γ_g-action on $\widetilde{S}'$ which is free and discontinuous, and the map D' is obviously ρ-equivariant. Hence, this defines a new marked projective structure with holonomy ρ: the grafting of σ over the graftable multi-curve α.

2.3. **Isotopy class of graftable curves.** It is an easy fact to verify that if α and α' belong to the same connected component of the set of graftable multi-curves (for the compact open topology), then the resulting projective structures $\mathrm{Gr}(\sigma, \alpha)$ and $\mathrm{Gr}(\sigma, \alpha')$ are equivalent, in the sense that there exists a diffeomorphism which respects the marking and which is projective in any chart of the respective projective structures. However, we will see that it can happen that α and α' are two graftable multi-curves that are isotopic as multi-curves, but such that their corresponding graftings are not equivalent.

2.4. **The graph of multi-graftings.** Let ρ be a representation from Γ_g to $\mathrm{PSL}(2, \mathbb{C})$. Let us define the graph of multi-graftings $MG(\rho)$ in the following way. The vertices are

the equivalence classes of marked projective structure with holonomy conjugated to ρ, modulo projective diffeomorphisms respecting the marking. The positive edges between two vertices (S_1, σ_1) and (S_2, σ_2) are the connected components of graftable multi-curves α in S_1 such that $\mathrm{Gr}(\sigma_1, \alpha) = \sigma_2$.

3. Fuchsian case: construction of graftable curves

Recall that a representation $\rho : \Gamma_g \to \mathrm{PSL}(2, \mathbb{R})$ is Fuchsian if it is discrete and faithful. In the sequel ρ will always be assumed to be Fuchsian.

3.1. **Goldman's parametrization of $MG(\rho)$.** We will denote by σ_u the uniformizing structure on the surface $S_u := \rho(\Gamma_g)\backslash\mathbb{H}^2$, which is obtained by taking the quotient of $\mathbb{H}^2$ by the ρ-action of Γ_g on $\mathbb{H}^2$. For this structure, the developing map is just the identity when identifying the universal cover of S_u with $\mathbb{H}^2$, and in particular is injective. Hence, any simple closed curve on S_u is a graftable curve. Moreover, by 2.3 the grafting $\mathrm{Gr}(\sigma_u, \alpha)$ depends only on the isotopy class of α as a multi-curve.

Goldman proved that every marked projective structure σ with holonomy ρ is obtained by grafting the structure σ_u along a multi-curve $\alpha = \{\alpha_i\}_i$. Moreover, this family is unique, and can be reconstructed from σ in the following way. For a Fuchsian projective structure σ, denote by $S^{\mathbb{R}}$ (resp. $S^{\pm}$) the quotient of $D^{-1}(\mathbb{RP}^1)$ (resp. $D^{-1}(\mathbb{H}^{\pm})$) by the covering group Γ_g. Since ρ is Fuchsian, it preserves the decomposition $\mathbb{CP}^1 = \mathbb{H}^+ \cup \mathbb{RP}^1 \cup \mathbb{H}^-$, and thus $S^{\mathbb{R}}$ is an analytic real submanifold of S separating S in domains which are either positive or negative according they belong to S^+ or S^-. Goldman proved that the components of S^- are necessarily annuli. The isotopy classes of those components define a multi-curve α which is such that $\sigma = \mathrm{Gr}(\sigma_u, \alpha)$. We denote in this case $\mathrm{Gr}_\alpha := \mathrm{Gr}(\sigma_u, \alpha)$.

3.2. **Homotopically transverse multi-curves.** Let $\alpha = \{\alpha_i\}_{i\in I}$ and $\beta = \{\beta_j\}_{j\in J}$ be two multi-curves. They are homotopically transverse if the following conditions hold:

- for each $i \in I$ and $j \in J$, the curves α_i and β_j are not homotopic
- they are transverse in the usual sense and
- any component of $\beta_j \cap (S \setminus \cup_i \alpha_i)$ sitting in a component C of $S \setminus \cup_i \alpha_i$ is not homotopically equivalent (fixing endpoints) in $\overline{C}$ to a segment in ∂C and the same remains true if we exchange the roles of the multicurves α and β.

3.3. **Construction of graftable multi-curves.** Given two homotopically transverse multicurves $\alpha = \{\alpha_i\}_{i\in I}$ and $\beta = \{\beta_j\}_{j\in J}$, assign to each of the curves α_i a turning direction $T_i \in \{R, L\}$ ("Right" or "Left") in such a way that any two parallel curves have the same turning direction. In this paragraph we provide a construction of a multi-curve β_T, which is graftable in Gr_α, corresponding to the data (α, β, T).

We begin by assuming that there are no parallel curves in the families α and β. In this case we can assume that the components of α and β are simple closed geodesics in the uniformizing structure σ_u.

We construct first the intersection of β_T with $S_u \setminus \alpha \subset S$, and then construct the intersection of β_T with the grafting annuli glued to $S_u \setminus \alpha$ to obtain Gr_α.

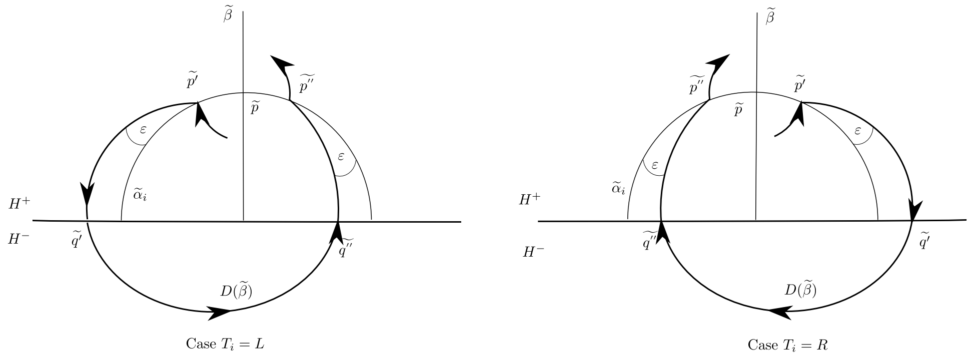


FIGURE 1. Finding graftable curves in $\mathbb{CP}^1 \setminus \overline{\widetilde{\alpha}_i}$

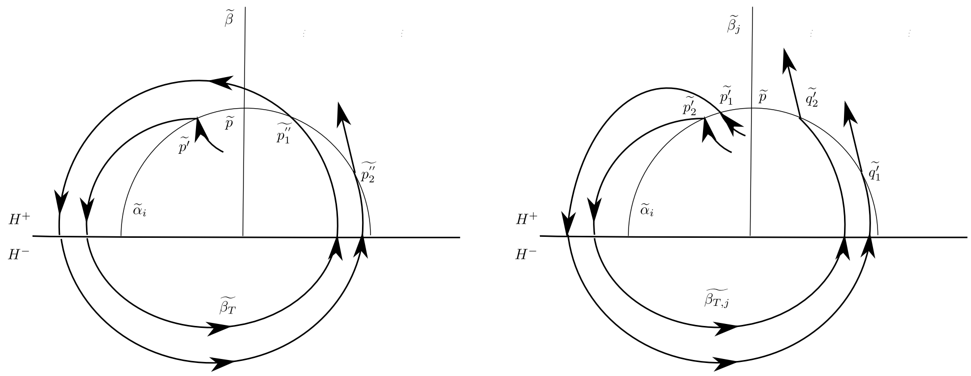
For each component C of $S_u \setminus \alpha$, its boundary is a union of α-components α_i. We fix a small positive number ε, and for each $p \in \alpha_i \cap \beta$, we consider the point $p_T \in \alpha_i$ lying at distance ε from p to the side of p indicated by $T(\alpha_i)$ with respect to the orientation induced on α_i by C. If we do this for all components of $S_u \setminus \alpha$, we get for each point $p \in \alpha_i \cap \beta$ a couple of distinct points $p', p'' \in \alpha_i$ lying at distance ε from p.

Now, the intersection of β with C is a union of geodesic segments $[p, q]$ joining points of ∂C. We define β_T in $S_u \setminus \alpha \subset S$ to be the union of the segments $[p', q']$ with p' and q' constructed as above. Observe that if we move the points p, q a little bit, then the segments $[p', q']$ are disjoint in the component C, but also in the whole surface S.

Then, one has to define the curve β_T in the grafting annuli. The continuation should start from the point p' above and end at p''. Figure 1 provides a sketch of a construction in the universal cover. When the path β_T enters in the grafting, it means that its lift $\widetilde{\beta_T}$ enters in the subset $\mathbb{CP}^1 \setminus \overline{\widetilde{\alpha}_i}$ that we have glued to $\widetilde{S_u}$ to obtain Gr_α. It enters at the point $\tilde{p'}$ and needs to get out at the point $\tilde{p''}$ by a path in $\mathbb{CP}^1 \setminus \overline{\widetilde{\alpha}_i}$. For this it has to turn around the segment $\overline{\widetilde{\alpha}_i}$ in the sphere. Since we want a graftable curve we avoid creating selfintersection points of the image of β_T. An example of such a curve can be constructed by taking the semi-infinite geodesic segments in $\mathbb{H}$ forming an angle of ε with $\widetilde{\alpha}_i$ at $\widetilde{p'}$ (respectively $\widetilde{p''}$) to the side determined by T on α_i (respectively opposite to that determnied by T). This produces two points $\widetilde{q'}, \widetilde{q''} \in \mathbb{RP}^1$ as their respective points at infinity, and they can be joined by the bi-infinite geodesic in $\mathbb{H}^-$ having them at infinity.

In this way we define a family of curves β_T which is homotopic to β. We claim that this family is indeed a multicurve graftable in Gr_α. There are two things to be verified: first that the curves $\hat{\beta}_j$ are embedded and disjoint. Second that the developing map of Gr_α is injective on each curve $\widetilde{\hat{\beta}_j}$. This holds as soon as ε is small enough. We leave the details for the reader.

We now explain the variation of the construction when some α_i appear with multiplicity or are parallel. As was said before, it is then very important that the same letter occurs for the corresponding parallel curves. If d_i is this multiplicity, then we consider a sequence of points $p_0 = p', p_1, \ldots, p_{d_i} = p''$ in $\widetilde{\alpha}_i$ increasing from p' to p'', and we do the similar



Case $T_i = L$ with α_i of multiplicity 2 Case $T_i = L$ with β_j of multiplicity 2

FIGURE 2. Graftable curves in the case of higher multiplicites

construction sketched in the left part of Figure 2 for the case of multiplicity $d_i = 2$ and $T_i = L$. The more general case can be easily sketched from that one.

If some of the β_j come with multiplicity e_j, then again we consider sequences of points $p'_{e_j} = p', \dots, p'_1, p''_1, \dots, p''_{e_j} = p''$ increasing from p' to p'' and we do the construction indicated in the right part of Figure 2 for the case of $T_i = L$ and β_j of multiplicity 2 (the general case is sketched similarly).

If both α_i and β_j come with multiplicities, we combine both variations in a suitable way.

Remark 3.1. *Note that in particular, we proved that, if σ is projective structure on a marked surface S with Fuchsian holonomy, and β is any multi-curve without component homotopic to a point, then it is possible to find a multi-curve which is graftable in σ and isotopic to β. It would be interesting to prove an analogous statement for a general projective structure (not necessarily with Fuchsian holonomy) and a general multi-curve β whose components have loxodromic holonomy.*

Remark 3.2. *There are other ways of finding graftable curves in the isotopy class of β, obtained by fixing a letter to each equivalence class of parallel curves of the multi-curve β, instead of α. However, this construction of multi-curve will not be discussed here.*

3.4. **Operations on homotopically transverse multi-curves.** Associated to (α, β, T) we produce a new isotopy class of a multi-curve γ in the hyperbolic surface S_u in the following way: at each point of intersection $p \in \alpha_i \cap \beta_j$ choose a disc D_p centered at p. After an isotopy we can suppose that this disc is parametrized by an orientation preserving map of the unit disc in the plane to S_u and the image of α_i corresponds to the horizontal axis and that of β_j to the vertical axis.

On $S_u \setminus \cup D_p$ the multi-curve γ has the same components as $\alpha \cup \beta$. To get a multi-curve we need to join the endpoints by paths on $\cup \partial D_p$ by the rule given by T: an endpoint of $\alpha_i \cap \partial D_p$ is joined through a simple curve in ∂D_p to the next point of $\beta_j \cap \partial D_p$ in the direction of ∂D_p given by the opposite of T_i (with respect to the orientation of ∂D_p). For an endpoint $\beta_j \cap \partial D_p$ where $p = \alpha_i \cap \beta_j$ we follow ∂D_p in the direction given by T_i until

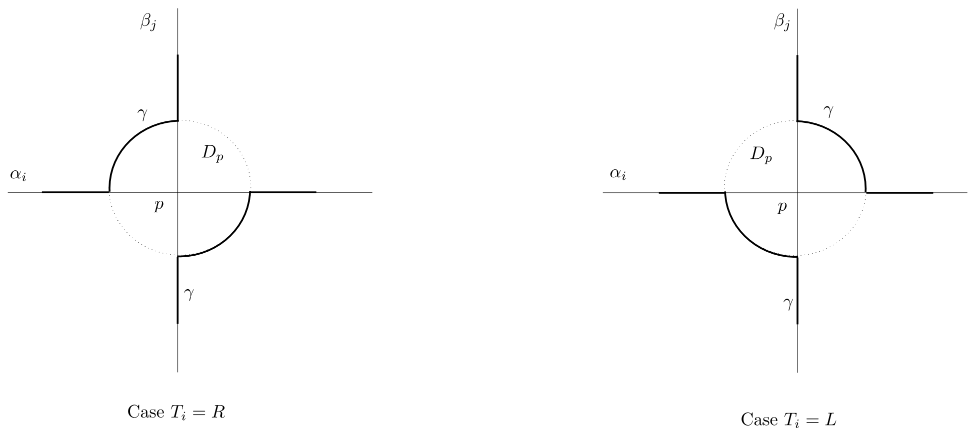


FIGURE 3. Construction of γ around a point of intersection between α_i and β_j

the following point in $\alpha_i \cap D_p$. This produces a family of disjoint simple closed curves γ in S_u. The transversality condition guarantees that none of its components is homotopically trivial in S_u and hence γ is a multi-curve (see figure 3). (See also [4, 5].)

This produces a family of disjoint simple closed curves that we denote by γ. The homotopical transversality hypothesis ensures that the components of γ are non homotopic to points. In the sequel, for given (α, β, T) we will also denote

$$\alpha *_T \beta := \gamma$$

3.5. **Computation of grafting annuli.** Remind that we use the notation $\mathrm{Gr}_\alpha = \mathrm{Gr}(\sigma_u, \alpha)$.

Proposition 3.3. *Given (α, β, T) as before let β_T denote the graftable multi-curve constructed in subsection 3.3 and let $\gamma = \alpha *_T \beta$. Then*

$$\mathrm{Gr}(\mathrm{Gr}_\alpha, \beta_T) = \mathrm{Gr}_\gamma.$$

Proof. We have to compute what are the negative annuli for the structure $\sigma' = \mathrm{Gr}(\mathrm{Gr}_\alpha, \beta_T)$ given by Goldman's theorem, see 3.1. To this end, we construct in each negative annuli a simple closed curve capturing its homotopy, such that the collection of these curves is a multi-curve isotopic to γ, hence proving the desired result.

To begin with, it is useful to orient each component $\beta_{T,j}$ of the path β_T. Let $\widetilde{\beta_{T,j}}$ be a lift of $\beta_{T,j}$. If (S, σ) denotes the projective surface corresponding to the structure $\sigma = \mathrm{Gr}_\alpha$, $\widetilde{S}$ is constructed by gluing to $\widetilde{S_u} \setminus \bigcup \widetilde{\alpha_i}$ the sets $\mathbb{CP}^1 \setminus \overline{\widetilde{\alpha_i}}$'s. In each of these sets, let $\widetilde{\alpha_i}^-$ be the geodesic in $\mathbb{H}^-$ which is the continuation of the geoedesic $\widetilde{\alpha_i}$ as a round circle of the Riemann sphere. The curves $\widetilde{\beta_{T,j}}$ intersect these geodesics successively. We denote them in order of intersection with $\widetilde{\beta_{T,j}}$ by $\widetilde{\alpha_n}^-$, for $n \in \mathbb{Z}$. For each n, we denote by $\tilde{r}_n$ the point of intersection of $\widetilde{\beta_{T,j}}$ and of $\widetilde{\alpha_n}^-$ (see Figure 4).

Observe that $\widetilde{\beta_T}$ cuts the curve $\widetilde{\alpha_n}^-$ as a right part $(\widetilde{\alpha_n})_R^-$ and a left part $(\widetilde{\alpha_n})_L^-$.

Let us now consider the projective surface $(S', \sigma') = \mathrm{Gr}(\sigma, \beta_T)$. The right and left parts of $\widetilde{\alpha_n}^-$ are cut by $\widetilde{\beta_T}$. We extend these curves in the bubble $\mathbb{CP}^1 \setminus \overline{D(\widetilde{\beta_T})}$ by the

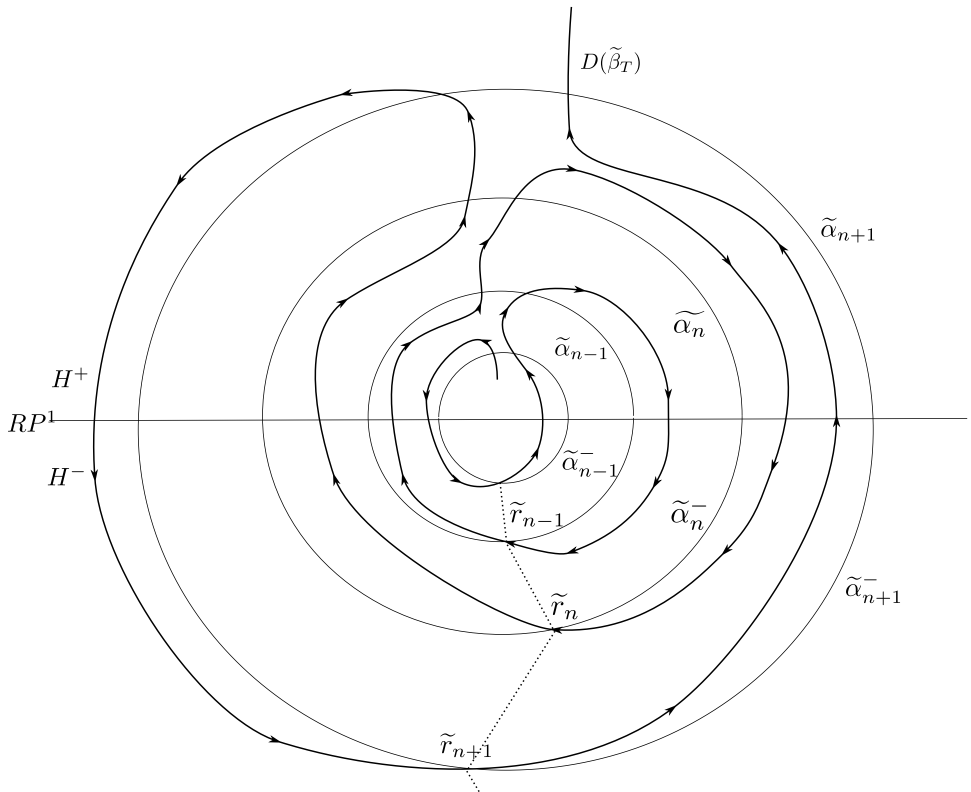
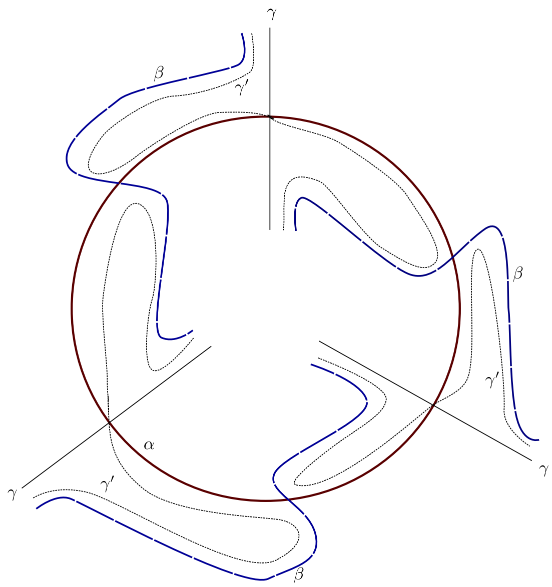


FIGURE 4. Detecting the grafting annuli

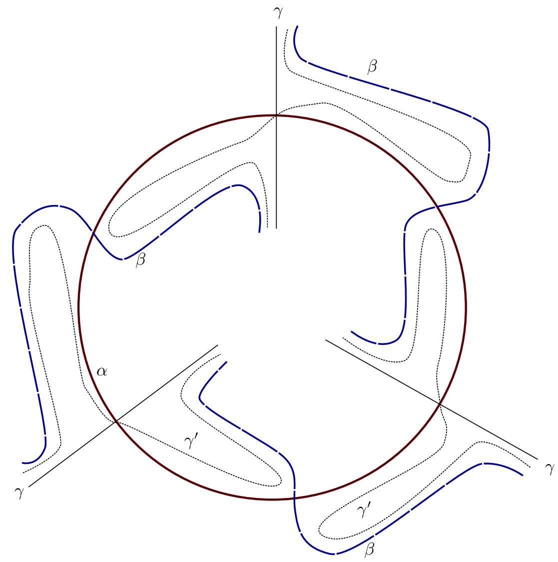
geodesic segments $(\tilde{r}_n, \tilde{r}_{n-1}) \subset \mathbb{CP}^1 \setminus \overline{D(\widetilde{\beta_T})}$ with respect to Poincaré metric in lower-half plane (observe that these geodesic segments are completely contained in the bubble as is shown in the figure and that one of the two geodesic segments $(\tilde{r}_n, \tilde{r}_{n-1})$ or $(\tilde{r}_n, \tilde{r}_{n+1})$ is the continuation of the left and right parts of $\widetilde{\alpha_n}^-$). This gives rise to a disjoint union of curves in $\widetilde{S}'$ which are included in the negative part.

Let us analyze more in detail this family. The right segment $(\widetilde{\alpha_n})^-_R$ of $\widetilde{\alpha_n}^-$ is continued by the geodesic segment $(\tilde{r}_n, \tilde{r}_{n-1})$ if the turning T_n corresponding to α_n is R. This is due to the fact that when the right segment enters in the bubble, then it crosses $\widetilde{\beta_{T,j}}$ and see the orientation of this later on the right. Hence, the continuation of this in the bubble has to be the segment which begins at $\tilde{r}_n$ and also sees $\widetilde{\beta_{j,T}}$ on the right. This segment has to be $(\tilde{r}_n, \tilde{r}_{n-1})$ if the turning T_n is R, see figure. If instead the turning T_n is L, then the same reasoning shows that it is continued by the geodesic segment $(\tilde{r}_n, \tilde{r}_{n-1})$.

Hence, the family of curves that we construct is isotopic to the family $\widetilde{\gamma}$. In particular they are not homotopic to points. Since every connected component of the negative part contains exactly one of these curves, the proof of the proposition follows. $\square$



Constructing β from γ and α when turning right (R)



Constructing β from γ and α when turning left (L)

FIGURE 5. Constructing β

4. Positive connectedness

In this part we prove theorem 1.1. We begin by the following:

Lemma 4.1. *Let α and γ be two multi-curves in S intersecting transversally in the sense of 3.3. Suppose that every component of α intersects γ and vice versa. Choose some letter for each equivalence class of parallel α-component. Then there exists a multi-curve β intersecting α transversally in the sense of 3.3 and such that, the resulting multi-curve obtained by the procedure explained in 3.3 is a multi-curve isotopic to γ.*

Proof. The proof is done by first constructing a multi-curve γ' isotopic to the multi-curve γ. For each component α_i of α, depending on its letter, deform γ in a small annular neighborhood of α_i as indicated in Figure 5, depending on the letter specified. Then define the multi-curve β as indicated in figure 5. It has the required properties. $\square$

We are now in a position to prove theorem 1.1. Let (S_i, σ_i), $i = 1, 2$, be projective structures with holonomy ρ, both different from the uniformizing structure σ_u. We denote by α_1 and α_2 the two multi-curves coding the negative annuli of σ_1 and σ_2 (that we think as a multi-geodesic with heights) so that $\sigma_i = \mathrm{Gr}_{\alpha_i}$. Consider a (single) simple closed geodesic γ cutting all components of α_1 and all components of α_2 (that a simple closed geodesic exists is left to the reader), and denote $(S_3, \sigma_3) = \mathrm{Gr}_\gamma$. By two applications of proposition 3.3 and lemma 4.1, there exist a multi-curve $\hat{\beta}_1 \subset S_1$ and a multi-curve $\hat{\beta}_2 \subset S_3$ such that $\mathrm{Gr}(\sigma_1, \hat{\beta}_1) = \sigma_3$ and $\mathrm{Gr}(\sigma_3, \hat{\beta}_2) = \sigma_2$. This proves the theorem.

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